

CONVEX BODIES AND MULTIPLICITIES OF IDEALS

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ABSTRACT. We associate convex regions in the positive orthant $\mathbb{R}_{\geq 0}^n$ to primary ideals in an n -dimensional regular local ring which encode information about the Samuel multiplicities of the ideals. More generally, we associate convex regions to graded sequences of primary ideals and in a larger class of local rings (including local rings of toroidal singularities). This is in the spirit of the theory of Gröbner bases and Newton polyhedra on one hand, and the theory of Newton-Okounkov bodies for linear systems on the other hand. We use this to give a new proof of a Brunn-Minkowski inequality for multiplicities of ideals due to Teissier and Rees-Sharp.

INTRODUCTION

The purpose of this note is to employ, in the local case, techniques from the theory of semigroups of integral points and Newton-Okounkov bodies (for the global case) and to obtain new results as well as new proofs of some previously known results about multiplicities of ideals in local rings.

Let $R = \mathbf{k}[x_1, \dots, x_n]_{(0)}$ be the algebra of polynomials localized at the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Let \mathfrak{a} be a primary ideal of R with a set of generators g_1, \dots, g_r (i.e. \mathfrak{a} contains a power of the maximal ideal \mathfrak{m}). Let f_1, \dots, f_n be n generic linear combinations of the g_i (with coefficients in \mathbf{k}). The *multiplicity* $e(\mathfrak{a})$ of the ideal \mathfrak{a} is the intersection multiplicity, at the origin, of the hypersurfaces $H_i = \{(x_1, \dots, x_n) \mid f_i(x_1, \dots, x_n) = 0\}$, $i = 1, \dots, n$. (It can be shown that this number is independent of the choice of the f_i and the generators g_j .) One shows that the multiplicity $e(\mathfrak{a})$ is equal to

$$n! \lim_{k \rightarrow \infty} \frac{\dim_{\mathbf{k}}(R/\mathfrak{a}^k)}{k^n}.$$

(This result is analogous to Hilbert's theorem on the Hilbert function and degree of a projective variety.) More generally, let R be a Noetherian local domain of dimension n over a base field \mathbf{k} . Let \mathfrak{a} be a primary ideal of R . Since \mathfrak{a} contains a power of the maximal ideal \mathfrak{m} , R/\mathfrak{a} is finite dimensional regarded as a vector space over \mathbf{k} . Geometrically speaking, if $R = \mathcal{O}_{X,x}$ is the local ring of a point x on a variety X over an algebraically closed field \mathbf{k} , an ideal \mathfrak{a} is primary if and only if it defines the single point x in a neighborhood of X . The *Hilbert-Samuel function* of

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the primary ideal \mathfrak{a} is defined by

$$H_{\mathfrak{a}}(k) = \dim_{\mathbf{k}}(R/\mathfrak{a}^k).$$

For large values of k , $H_{\mathfrak{a}}(k)$ coincides with a polynomial of degree n called the *Hilbert-Samuel polynomial* of \mathfrak{a} . The *Samuel multiplicity*, $e(\mathfrak{a})$ of \mathfrak{a} is defined to be the leading coefficient of $H_{\mathfrak{a}}(k)$ multiplied by $n!$.

It is well-known that the Samuel multiplicity satisfies a Brunn-Minkowski inequality [Teissier77, Rees-Sharp78]. That is, for any two primary ideals $\mathfrak{a}, \mathfrak{b} \in R$ we have

$$(1) \quad e(\mathfrak{a}\mathfrak{b})^{1/n} \leq e(\mathfrak{a})^{1/n} + e(\mathfrak{b})^{1/n}.$$

More generally, one can define Samuel multiplicity for a primary graded sequence of ideals. A *graded sequence of ideals* is a sequence $\mathfrak{a}_{\bullet} = (\mathfrak{a}_1, \mathfrak{a}_2, \dots)$ of ideals in R such that for all k, ℓ we have $\mathfrak{a}_k \mathfrak{a}_{\ell} \subset \mathfrak{a}_{k+\ell}$. A graded sequence of ideals \mathfrak{a}_{\bullet} is *primary* if \mathfrak{a}_1 contains a power of the maximal ideal \mathfrak{m} . It follows that for any k , R/\mathfrak{a}_k is finite dimensional over \mathbf{k} . The multiplicity $e(\mathfrak{a}_{\bullet})$ is defined to be

$$(2) \quad \limsup_k \frac{\dim_{\mathbf{k}}(R/\mathfrak{a}_k)}{k^n}.$$

(It is not a priori clear that the limit exists.)

We will use convex geometric arguments to prove the existence of the limit in (2) and the inequality (1) for a large class of local domains R . This class contains regular local rings (i.e. local rings of smooth points on algebraic varieties) and more generally the local rings of toroidal singularities (i.e. a singular point on an algebraic variety which is analytically isomorphic to a neighborhood in a toric variety). To make the presentation simpler, we mostly focus on the case where R is a regular local ring. In this case we will deal with convex subsets of the positive orthant $\mathbb{R}_{\geq 0}^n$. We call a closed convex subset $\Gamma \subset \mathbb{R}_{\geq 0}^n$, an $\mathbb{R}_{\geq 0}^n$ -*convex region* if the following condition holds: for any $x \in \Gamma$ we have $x + \mathbb{R}_{\geq 0}^n \subset \Gamma$. To a graded sequence of ideals \mathfrak{a}_{\bullet} we associate a convex region $\Gamma(\mathfrak{a}_{\bullet}) \subset \mathbb{R}_{\geq 0}^n$, such that when \mathfrak{a}_{\bullet} is primary the set $\mathbb{R}_{\geq 0}^n \setminus \Gamma(\mathfrak{a}_{\bullet})$ is bounded (Definition 6.7). The main result of the note (Theorem 6.8) is that the limit in (2) exists and

$$e(\mathfrak{a}_{\bullet}) = n! \operatorname{vol}(\mathbb{R}_{\geq 0}^n \setminus \Gamma(\mathfrak{a}_{\bullet})).$$

The construction of $\Gamma(\mathfrak{a}_{\bullet})$ is an analogue of the construction of the Newton-Okounkov body of a linear system on an algebraic variety (see [Okounkov03], [Okounkov96], [Kaveh-Khovanskii09], [Lazarsfeld-Mustata08]).

On the other hand, the construction of $\Gamma(\mathfrak{a})$ generalizes the notion of the Newton diagram of a power series (see [Arnold-Varchenko-Guseinzade85, Section 12.7]). To a monomial ideal in a polynomial ring (or a power series ring), i.e. an ideal generated by monomials, one can associate its (unbounded) *Newton polyhedron*. It is the convex hull of the exponents of the monomials appearing in the ideal. The *Newton diagram* of a monomial ideal is the union of the bounded faces of the Newton polyhedron. One can see that for a monomial ideal \mathfrak{a} , the convex region $\Gamma(\mathfrak{a})$ coincides with its Newton polyhedron (Proposition 7.2). The main theorem in this manuscript (Theorem 6.8) for the case of monomial ideals recovers the local case of the well-known theorem of Bernstein-Kushnirenko, about computing the multiplicity at the origin of a system $f_1(x) = \dots = f_n(x) = 0$ where the f_i are generic functions

from primary monomial ideals (see Section 7 and [Arnold-Varchenko-Guseinzade85, Section 12.7]).

More generally, let C be a closed strictly convex cone with apex at the origin (i.e. C is a convex cone and does not contain any line). We call a closed convex set $\Gamma \subset C$, a C -convex region if for any $x \in \Gamma$ we have $x + C \subset \Gamma$. We say that Γ is *cobounded* (alternatively Γ is *inscribed* in C) if $C \setminus \Gamma$ is bounded. It is easy to verify that the set of cobounded C -convex regions is closed under addition (Minkowski sum of convex sets) and multiplication with a positive real number. For a cobounded C -convex region Γ we call the volume of the bounded region $C \setminus \Gamma$ the *covolume* of Γ and denote it by $\text{covol}(\Gamma)$. Similar to convex bodies and their volumes and mixed volumes, one can develop a theory of cobounded convex regions and their covolumes and mixed covolumes. In [Khovanskii-Timorin2012] the authors prove an analogue of the Alexandrov-Fenchel inequality for the covolumes of convex regions (see Theorem 2.3). The usual Alexandrov-Fenchel inequality is an important inequality about volumes of convex bodies in \mathbb{R}^n and generalizes the classical isoperimetric inequality and the Brunn-Minkowski inequality. In a similar way, the result in [Khovanskii-Timorin2012] implies a Brunn-Minkowski inequality for covolumes, that is, for any two cobounded C -convex regions Γ_1, Γ_2 where C is an n -dimensional cone, we have:

$$\text{covol}(\Gamma_1)^{1/n} + \text{covol}(\Gamma_2)^{1/n} \geq \text{covol}(\Gamma_1 + \Gamma_2)^{1/n}.$$

To associate a convex region to an ideal we need a valuation on the ring R . We will assume that there is a valuation v on R with values in \mathbb{Z}^n (with respect to some total order on \mathbb{Z}^n) such that: (1) The residue field of v is \mathbf{k} , and (2) the value semigroup consists of all the integral points in a strongly convex rational polyhedral convex cone C . As an example, let $R = \mathbf{k}[x_1, \dots, x_n]_{(0)}$ as before be the algebra of polynomials localized at the maximal ideal (x_1, \dots, x_n) . Then the map which associates to a polynomial its lowest exponent (with respect to some lexicographic order) defines a valuation on R and the value semigroup coincides with the semigroup $\mathbb{Z}_{\geq 0}^n$, that is, the semigroup of all the integral points in the positive orthant $\mathbb{R}_{\geq 0}^n$. The above condition is satisfied for the local ring of any smooth point on an algebraic variety and more generally the local ring of any point with a toroidal singularity.

As an application of the construction of $\Gamma(\mathfrak{a})$, we apply the Brunn-Minkowski inequality for convex regions to obtain a convex geometric proof of the inequality (1) (Corollary 6.10). We prove all the main results in this note for the more general setting of graded sequences of ideals.

The Brunn-Minkowski inequality proved in this paper is a corollary of the more general Alexandrov-Fenchel inequality for mixed multiplicities of ideals. Take primary ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ in a local ring $R = \mathcal{O}_{X,x}$ of a point x on an n -dimensional algebraic variety X . The *mixed multiplicity* $e(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ is equal to the intersection multiplicity, at the origin, of the hypersurfaces $H_i = \{x \mid f_i(x) = 0\}$, $i = 1, \dots, n$, where each f_i is a generic function from the ideal \mathfrak{a}_i , i.e. f_i is a generic linear combination (with coefficients in \mathbf{k}) of a system of generators of \mathfrak{a}_i . Alternatively one can define the mixed multiplicity as the polarization of the Hilbert-Samuel multiplicity $e(\mathfrak{a})$, i.e. it is the unique function $e(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ which is invariant under permuting the arguments, is multi-additive with respect to product of ideals, and for any primary ideal \mathfrak{a} the mixed multiplicity $e(\mathfrak{a}, \dots, \mathfrak{a})$ coincides with $e(\mathfrak{a})$.

The Alexandrov-Fenchel inequality is the following inequality among the mixed multiplicities of the \mathfrak{a}_i :

$$(3) \quad e(\mathfrak{a}_1, \mathfrak{a}_1, \mathfrak{a}_3, \dots, \mathfrak{a}_n) e(\mathfrak{a}_2, \mathfrak{a}_2, \mathfrak{a}_3, \dots, \mathfrak{a}_n) \geq e(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \dots, \mathfrak{a}_n)^2$$

Our technique of associating convex regions to ideals in this manuscript does not give us a proof of the Alexandrov-Fenchel inequality (3). The reason is that in order to prove the general case we need to deal with local rings of more general kinds of singularities (rather than smooth or toroidal). In a coming paper we give a simple proof of the Alexandrov-Fenchel inequality for mixed multiplicities of ideals using arguments similar to those of this paper. This proof then implies an Alexandrov-Fenchel inequality for covolumes of convex regions (in a similar way that in [Kaveh-Khovanskii09] and in [Khovanskii88] the authors obtain a proof of the usual Alexandrov-Fenchel inequality for volumes of convex bodies). It is closely related to the results in [Teissier77, Rees-Sharp78].

When this note was near completion we learned about [Cutkosky12] which also uses techniques from the theory of Newton-Okounkov bodies and applies it to the more general notion of epsilon multiplicity of an ideal.

And few words about the organization of the paper: Section 1 recalls basic background material about volumes/mixed volumes of convex bodies. Section 2 is about convex regions and their covolumes/mixed covolumes, which we can think of as a local version of the theory of mixed volumes of convex bodies. In Sections 3 and 4 we associate a convex region to a semigroup ideal of integral points and prove the main combinatorial result required later (Definitions 4.2, 4.7 and Theorem 4.10). Finally in Section 6, using a valuation on the ring R , we associate a convex region $\Gamma(\mathfrak{a}_\bullet)$ to a graded sequence of ideals \mathfrak{a}_\bullet and prove the main results of this note (Theorem 6.8 and Corollary 6.10). The last section (Section 7) discusses the case of monomial ideals and the Bernstein-Kushnirenko theorem.

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1. MIXED VOLUME OF CONVEX BODIES

The collection of all convex bodies in \mathbb{R}^n is a cone, that is, we can add convex bodies and multiply a convex body with a positive number. For two convex bodies $\Delta_1, \Delta_2 \subset \mathbb{R}^n$, their (Minkowski) sum $\Delta_1 + \Delta_2$ is $\{x + y \mid x \in \Delta_1, y \in \Delta_2\}$. Let vol denote the n -dimensional volume in \mathbb{R}^n with respect to the standard Euclidean metric. The function vol is a homogeneous polynomial of degree n on the cone of convex bodies, i.e. its restriction to each finite dimensional section of the cone is a homogeneous polynomial of degree n . In other words, for any collection of convex bodies $\Delta_1, \dots, \Delta_r$, the function:

$$P_{\Delta_1, \dots, \Delta_r}(\lambda_1, \dots, \lambda_r) = \text{vol}(\lambda_1 \Delta_1 + \dots + \lambda_r \Delta_r),$$

is a homogeneous polynomial of degree n in $\lambda_1, \dots, \lambda_r$. By definition the *mixed volume* $V(\Delta_1, \dots, \Delta_n)$ of an n -tuple $(\Delta_1, \dots, \Delta_n)$ of convex bodies is the coefficient of the monomial $\lambda_1 \cdots \lambda_n$ in the polynomial $P_{\Delta_1, \dots, \Delta_n}(\lambda_1, \dots, \lambda_n)$ divided by $n!$. This definition implies that mixed volume is the *polarization* of the volume polynomial, that is, it is the unique function on the n -tuples of convex bodies satisfying the following:

- (i) (Symmetry) V is symmetric with respect to permuting the bodies $\Delta_1, \dots, \Delta_n$.
- (ii) (Multi-linearity) It is linear in each argument with respect to the Minkowski sum. The linearity in first argument means that for convex bodies $\Delta'_1, \Delta''_1, \Delta_2, \dots, \Delta_n$, and real numbers $\lambda', \lambda'' \geq 0$ we have:

$$V(\lambda' \Delta'_1 + \lambda'' \Delta''_1, \dots, \Delta_n) = \lambda' V(\Delta'_1, \dots, \Delta_n) + \lambda'' V(\Delta''_1, \dots, \Delta_n).$$
- (iii) (Relation with volume) On the diagonal it coincides with the volume, i.e. if $\Delta_1 = \dots = \Delta_n = \Delta$, then $V(\Delta_1, \dots, \Delta_n) = \text{vol}(\Delta)$.

The following inequality attributed to Alexandrov and Fenchel is important and very useful in convex geometry:

Theorem 1.1 (Alexandrov-Fenchel). *Let $\Delta_1, \dots, \Delta_n$ be convex bodies in \mathbb{R}^n . Then*

$$V(\Delta_1, \Delta_1, \Delta_3, \dots, \Delta_n) V(\Delta_2, \Delta_2, \Delta_3, \dots, \Delta_n) \leq V^2(\Delta_1, \Delta_2, \dots, \Delta_n).$$

In dimension 2, this inequality is elementary. We will call it the *generalized isoperimetric inequality*, because when Δ_2 is the unit ball it coincides with the classical isoperimetric inequality. The celebrated *Brunn-Minkowski inequality* concerns volume of convex bodies in \mathbb{R}^n . It is an easy corollary of the Alexandrov-Fenchel inequality.

Theorem 1.2 (Brunn-Minkowski). *Let Δ_1, Δ_2 be convex bodies in \mathbb{R}^d . Then*

$$\text{vol}^{1/n}(\Delta_1) + \text{vol}^{1/n}(\Delta_2) \leq \text{vol}^{1/n}(\Delta_1 + \Delta_2).$$

2. MIXED COVOLUME OF CONVEX REGIONS

Let C be a strongly convex closed n -dimensional cone in \mathbb{R}^n with apex at the origin. We are interested in convex subsets of C which have bounded complement.

Definition 2.1. We call a closed convex subset $\Gamma \subset C$ a *C -convex region* (or simply a convex region when the cone C is understood from the context) if for any $x \in \Gamma$ and $y \in C$ we have $x + y \in \Gamma$. Moreover, we call a convex region Γ , *cobounded* (alternatively we say Γ is *inscribed* in C) if the complement $C \setminus \Gamma$ is bounded. In this case the volume of $C \setminus \Gamma$ is finite which we call the *covolume* of Γ and denote it by $\text{covol}(\Gamma)$.

The collection of C -convex regions (respectively cobounded regions) is closed under the Minkowski sum and multiplication by positive scalars. Similar to the volume of convex bodies, one proves that the covolume of convex regions is a homogeneous polynomial [Khovanskii-Timorin2012]. More precisely:

Theorem 2.2. *Let $\Gamma_1, \dots, \Gamma_r$ be cobounded C -convex regions in the cone C . Then the function:*

$$P_{\Gamma_1, \dots, \Gamma_r}(\lambda_1, \dots, \lambda_r) = \text{covol}(\lambda_1 \Gamma_1 + \dots + \lambda_r \Gamma_r),$$

is a homogeneous polynomial of degree n .

As in the case of convex bodies, one uses the above theorem to define mixed covolume of cobounded regions. By definition the *mixed covolume* $V(\Gamma_1, \dots, \Gamma_n)$ of an n -tuple $(\Gamma_1, \dots, \Gamma_n)$ of cobounded convex regions is the coefficient of the monomial $\lambda_1 \dots \lambda_n$ in the polynomial $P_{\Gamma_1, \dots, \Gamma_n}(\lambda_1, \dots, \lambda_n)$ divided by $n!$. That is, mixed covolume is the unique function on the n -tuples of cobounded regions satisfying the following:

- (i) (Symmetry) V is symmetric with respect to permuting the regions $\Gamma_1, \dots, \Gamma_n$.
- (ii) (Multi-linearity) It is linear in each argument with respect to the Minkowski sum.
- (iii) (Relation with covolume) For any cobounded region $\Gamma \subset C$:

$$V(\Gamma, \dots, \Gamma) = \text{covol}(\Gamma).$$

The mixed covolume satisfies an Alexandrov-Fenchel inequality [Khovanskii-Timorin2012]. Note that the inequality is reversed compared to the Alexandrov-Fenchel for mixed volumes of convex bodies.

Theorem 2.3 (Alexandrov-Fenchel for mixed covolume). *Let $\Gamma_1, \dots, \Gamma_n$ be cobounded C -convex regions in a cone C . Then*

$$V(\Gamma_1, \Gamma_1, \Gamma_3, \dots, \Gamma_n) V(\Gamma_2, \Gamma_2, \Gamma_3, \dots, \Gamma_n) \geq V^2(\Gamma_1, \Gamma_2, \dots, \Gamma_n).$$

The reversed Alexandrov-Fenchel inequality implies a reversed Brunn-Minkowski inequality:

Theorem 2.4 (Brunn-Minkowski for covolumes). *Let Γ_1, Γ_2 be cobounded C -convex regions in a convex cone C . Then*

$$\text{covol}^{1/n}(\Gamma_1) + \text{covol}^{1/n}(\Gamma_2) \geq \text{covol}^{1/n}(\Gamma_1 + \Gamma_2).$$

3. SEMIGROUPS OF INTEGRAL POINTS

In this section we recall some general facts from [Kaveh-Khovanskii09] about the asymptotic behavior of semigroups of integral points. Let $S \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$ be a semigroup. For simplicity, assume $S_1 \neq \emptyset$ and that S generates the whole lattice \mathbb{Z}^{n+1} . Let $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ denote the projection onto the first factor, and let $S_k = S \cap \pi^{-1}(k)$ be the set of points in S at level k . Define the function H_S by:

$$H_S(k) = \#S_k.$$

We call H_S the *Hilbert function of the semigroup S* . We wish to describe the asymptotic behavior of H_S as $k \rightarrow \infty$.

Let $C(S)$ be the closure of the convex hull of $S \cup \{0\}$, that is, the smallest closed convex cone (with apex at the origin) containing S . We call the projection of the convex set $C(S) \cap \pi^{-1}(1)$ to \mathbb{R}^n (under the projection onto the first factor $(x, 1) \mapsto x$), the *Newton-Okounkov convex set of the semigroup S* and denote it by $\Delta(S)$. In other words,

$$\Delta(S) = \overline{\text{conv}\left(\bigcup_{k>0} \{x/k \mid (k, x) \in S_k\}\right)}.$$

If $C(S) \cap \pi^{-1}(0) = \{0\}$ then $\Delta(S)$ is compact and hence a convex body.

The Newton-Okounkov convex set $\Delta(S)$ is responsible for the asymptotic behavior of the Hilbert function of S (see [Kaveh-Khovanskii09, Corollary 1.16]):

Theorem 3.1. *The limit*

$$\lim_{k \rightarrow \infty} \frac{H_S(k)}{k^n},$$

exists and is equal to $\text{vol}(\Delta(S))$.

4. SEMIGROUP IDEALS AND CONVEX REGIONS

In this section we discuss semigroup ideals of integral points and describe their asymptotic behavior using Theorem 3.1. A semigroup ideal of integral points, is an analogue of an ideal in a ring, for the semigroup $\mathbb{Z}_{\geq 0}^n$. In fact, the set of values of a valuation on an ideal in a ring is a semigroup ideal. In section 6 we will employ this to describe the asymptotic behavior of the Hilbert-Samuel function of a primary ideal in a local ring.

Let $\mathbb{R}_{\geq 0}^n = \{(x_1, \dots, x_n) \mid x_i \geq 0\}$ denote the positive orthant in \mathbb{R}^n , also let $\mathbb{Z}_{\geq 0}^n$ denote the semigroup of integral points contained in the positive orthant $\mathbb{R}_{\geq 0}^n$.

Remark 4.1. More generally, with slight modification the definitions and arguments in this section can be carried out for any strongly convex rational polyhedral cone C instead of $\mathbb{R}_{\geq 0}^n$ and the semigroup $C \cap \mathbb{Z}^n$ instead of $\mathbb{Z}_{\geq 0}^n$.

Definition 4.2. A subset $\mathcal{I} \subset \mathbb{Z}_{\geq 0}^n$ is called a *semigroup ideal* if for any $x \in \mathcal{I}$ and $y \in \mathbb{Z}_{\geq 0}^n$ we have $x + y \in \mathcal{I}$.

For two semigroup ideals \mathcal{I}, \mathcal{J} , the sum $\mathcal{I} + \mathcal{J} = \{x + y \mid x \in \mathcal{I}, y \in \mathcal{J}\}$ is again a semigroup ideal. For any integer $k > 0$, the product $k * \mathcal{I} = \mathcal{I} + \dots + \mathcal{I}$ (k times) is also a semigroup ideal.

Let $\mathcal{M} = \mathbb{Z}_{\geq 0}^n \setminus \{0\}$. Then \mathcal{M} is the unique maximal semigroup ideal in $\mathbb{Z}_{\geq 0}^n$, that is, any proper semigroup ideal \mathcal{I} is contained in \mathcal{M} .

Definition 4.3. We call a semigroup ideal \mathcal{I} *primary* if $\mathbb{Z}_{\geq 0}^n \setminus \mathcal{I}$ is finite. Equivalently \mathcal{I} is primary if for some integer $m > 0$, we have $m * \mathcal{M} \subset \mathcal{I}$. Clearly, if \mathcal{I} is primary then for any integer $k > 0$, $k * \mathcal{I}$ is also primary.

Definition 4.4. A *graded sequence of semigroup ideals* is a sequence $\mathcal{I}_{\bullet} = (\mathcal{I}_1, \mathcal{I}_2, \dots)$ of semigroup ideals such that for all k, ℓ we have $\mathcal{I}_k + \mathcal{I}_{\ell} \subset \mathcal{I}_{k+\ell}$. A graded sequence of semigroup ideals \mathcal{I}_{\bullet} is *primary*, if there is an integer $m > 0$ with $m * \mathcal{M} \subset \mathcal{I}_1$. It follows that for any k , $(km) * \mathcal{M} \subset \mathcal{I}_k$ and hence $\mathbb{Z}_{\geq 0}^n \setminus \mathcal{I}_k$ is finite.

Example 4.5. Let \mathcal{I} be a semigroup ideal. Then the sequence \mathcal{I}_{\bullet} defined by $\mathcal{I}_k = k * \mathcal{I}$ is clearly a graded sequence of semigroup ideals. \mathcal{I} is primary if and only if \mathcal{I}_{\bullet} is primary.

Let $\mathcal{I}'_{\bullet}, \mathcal{I}''_{\bullet}$ be graded sequences of semigroup ideals. Then the sequence $\mathcal{I}_{\bullet} = \mathcal{I}'_{\bullet} + \mathcal{I}''_{\bullet}$ defined by

$$\mathcal{I}_k = \mathcal{I}'_k + \mathcal{I}''_k,$$

is also a graded sequence of semigroup ideals which we call the *sum of sequences* \mathcal{I}'_{\bullet} and \mathcal{I}''_{\bullet} .

Definition 4.6. Let \mathcal{I}_{\bullet} be a primary sequence of semigroup ideals. Define the function $H_{\mathcal{I}_{\bullet}}$ by:

$$H_{\mathcal{I}_{\bullet}}(k) = \#(\mathbb{Z}_{\geq 0}^n \setminus \mathcal{I}_k).$$

We call it the *Hilbert-Samuel function of \mathcal{I}_{\bullet}* .

To a graded sequence of semigroup ideals \mathcal{I}_{\bullet} we can associate an $\mathbb{R}_{\geq 0}^n$ -convex region $\Gamma(\mathcal{I}_{\bullet})$ (see Definition 2.1). This convex set encodes information about the asymptotic behavior of the Hilbert-Samuel function of \mathcal{I}_{\bullet} .

Definition 4.7. Define the convex set $\Gamma(\mathcal{I}_\bullet)$ by

$$\Gamma(\mathcal{I}_\bullet) = \overline{\text{conv}\left(\bigcup_{k>0} \{x/k \mid x \in \mathcal{I}_k\}\right)}.$$

It is an unbounded convex set in $\mathbb{R}_{\geq 0}^n$.

From the definition it easily follows that:

Proposition 4.8. $\Gamma(\mathcal{I}_\bullet)$ is an $\mathbb{R}_{\geq 0}^n$ -convex region in the cone $\mathbb{R}_{\geq 0}^n$, i.e. if $x \in \Gamma(\mathcal{I}_\bullet)$ and $y \in \mathbb{R}_{\geq 0}^n$ then $x + y \in \Gamma(\mathcal{I}_\bullet)$. Moreover, if \mathcal{I}_\bullet is primary, the region $\Gamma(\mathcal{I}_\bullet)$ is cobounded i.e. $\mathbb{R}_{\geq 0}^n \setminus \Gamma(\mathcal{I}_\bullet)$ is bounded.

The following additivity property is straight forward from definition.

Proposition 4.9. Let $\mathcal{I}'_\bullet, \mathcal{I}''_\bullet$ be sequences of semigroup ideals. We have:

$$\Gamma(\mathcal{I}'_\bullet) + \Gamma(\mathcal{I}''_\bullet) = \Gamma(\mathcal{I}'_\bullet + \mathcal{I}''_\bullet).$$

The following is our main result about the asymptotic behavior of a graded sequence of semigroup ideals.

Theorem 4.10. Let \mathcal{I}_\bullet be a primary sequence of semigroup ideals. Then

$$\lim_{k \rightarrow \infty} \frac{H_{\mathcal{I}_\bullet}(k)}{k^n}$$

exists and is equal to $\text{covol}_n(\Gamma(\mathcal{I}_\bullet))$.

Definition 4.11. For a primary sequence \mathcal{I}_\bullet of semigroup ideals, we denote the limit $\lim_{k \rightarrow \infty} H_{\mathcal{I}_\bullet}(k)/k^n$ by $e(\mathcal{I}_\bullet)$. Motivated by commutative algebra, we call it the multiplicity of \mathcal{I}_\bullet . The multiplicity $e(\mathcal{I})$ of a primary semigroup ideal is the multiplicity of its associated sequence \mathcal{I}_\bullet defined by $\mathcal{I}_k = k * \mathcal{I}$.

Proof of Theorem 4.10. Choose m large enough so that $m * \mathcal{M}$ is contained in \mathcal{I}_1 , and moreover, the finite set $\mathcal{I}_1 \setminus m * \mathcal{M}$ generates the lattice \mathbb{Z}^n . Consider

$$S = \{(x, k) \mid x \in \mathcal{I}_k \setminus (km * \mathcal{M})\}.$$

$$T = \{(x, k) \mid x \in \mathbb{Z}_{\geq 0}^n \setminus (km * \mathcal{M})\}.$$

S and T are semigroups in $\mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$ and we have $S \subset T$. From the definition it follows that both of the groups generated by S and T are \mathbb{Z}^{n+1} . Also the Newton-Okounkov bodies of S and T are:

$$\Delta(S) = \Gamma(\mathcal{I}_\bullet) \cap \Delta(m),$$

$$\Delta(T) = \Delta(m),$$

where $\Delta(m) = \{(x_1, \dots, x_n) \mid x_1 + \dots + x_n \leq m\}$. Since $(km) * \mathcal{M} \subset \mathcal{I}_k$, for all k , we have:

$$\mathbb{Z}_{\geq 0}^n \setminus \mathcal{I}_k = T_k \setminus S_k.$$

(As usual $S_k = \{x \mid (x, k) \in S\}$ (respectively T_k) denotes the set of points in S (respectively T) at level k .) Hence

$$H_{\mathcal{I}_\bullet}(k) = \#(T_k) - \#(S_k).$$

By Theorem 3.1 we have:

$$\lim_{k \rightarrow \infty} \frac{\#S_k}{k^n} = \text{vol}_n(\Delta(S)),$$

$$\lim_{k \rightarrow \infty} \frac{\#(T_k)}{k^n} = \text{vol}_n(\Delta(m)).$$

Thus

$$\lim_{k \rightarrow \infty} \frac{\#(\mathbb{Z}_{\geq 0}^n \setminus \mathcal{I}_k)}{k^n} = \text{vol}_n(\Delta(m)) - \text{vol}_n(\Delta(S)).$$

On the other hand, we have:

$$\Delta(m) \setminus \Delta(S) = \mathbb{R}_{\geq 0}^n \setminus \Gamma(\mathcal{I}_{\bullet}),$$

and hence $\text{vol}_n(\Delta(m)) - \text{vol}_n(\Delta(S)) = \text{covol}_n(\Gamma(\mathcal{I}_{\bullet}))$. This finishes the proof. \square

Let $\mathcal{I}_1, \dots, \mathcal{I}_r$ be primary semigroup ideals in $\mathbb{Z}_{\geq 0}^n$. Define the function $P_{\mathcal{I}_1, \dots, \mathcal{I}_r} : \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{N}$ by:

$$P_{\mathcal{I}_1, \dots, \mathcal{I}_r}(k_1, \dots, k_r) = e(k_1 * \mathcal{I}_1 + \dots + k_r * \mathcal{I}_r).$$

Theorem 4.12. *The function $P_{\mathcal{I}_1, \dots, \mathcal{I}_r}$ is a homogeneous polynomial of degree n in k_1, \dots, k_r .*

Proof. Follows immediately from Proposition 4.9, Theorem 4.10 and Theorem 2.2. \square

From Theorem 4.10 and Proposition 4.9 we have the following corollary:

Corollary 4.13.

$$e(\mathcal{I}_1, \dots, \mathcal{I}_n) = n!V(\Gamma(\mathcal{I}_1), \dots, \Gamma(\mathcal{I}_n)),$$

where V denotes the mixed volume of cobounded regions.

From Theorem 2.3 we obtain:

Corollary 4.14 (Alexandrov-Fenchel inequality for mixed multiplicity of semigroup ideals). *For primary semigroup ideals $\mathcal{I}_1, \dots, \mathcal{I}_n \subset \mathbb{Z}_{\geq 0}^n$ we have:*

$$e(\mathcal{I}_1, \mathcal{I}_1, \mathcal{I}_3, \dots, \mathcal{I}_n) e(\mathcal{I}_2, \mathcal{I}_2, \mathcal{I}_3, \dots, \mathcal{I}_n) \geq e(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \dots, \mathcal{I}_n)^2.$$

5. MULTIPLICITIES OF IDEALS IN LOCAL RINGS

Let R be a Noetherian local domain of dimension n over a field \mathbf{k} , and with maximal ideal \mathfrak{m} . We also assume that the residue field R/\mathfrak{m} is \mathbf{k} .

Example 5.1. Let X be an irreducible variety over an algebraically closed field \mathbf{k} of dimension n , and let Z be an irreducible subvariety of X . Then the local ring $\mathcal{O}_{X,Z}$ of rational functions on X which are regular in a neighborhood of Z is a Noetherian local domain of dimension n over \mathbf{k} . The ideal \mathfrak{m} consists of functions which vanish on the subvariety Z .

An ideal \mathfrak{a} is called *primary* if \mathfrak{a} contains a power of the maximal ideal \mathfrak{m} . If \mathfrak{a} is primary then for any integer $k > 0$, R/\mathfrak{a}^k is a finite dimensional \mathbf{k} -vector space.

Example 5.2. An ideal \mathfrak{a} in $\mathcal{O}_{X,Z}$ is primary if its zero locus coincides with the subvariety Z .

Definition 5.3. A *graded sequence of ideals* is a sequence $\mathfrak{a}_{\bullet} = (\mathfrak{a}_1, \mathfrak{a}_2, \dots)$ of ideals such that for all k, ℓ we have $\mathfrak{a}_k \mathfrak{a}_{\ell} \subset \mathfrak{a}_{k+\ell}$. A graded sequence of ideals \mathfrak{a}_{\bullet} is *primary*, if there is an integer $m > 0$ with $\mathfrak{m}^m \subset \mathfrak{a}_1$. It follows that for any k , $\mathfrak{m}^{km} \subset \mathfrak{a}_k$ and hence R/\mathfrak{a}_k is finite dimensional over \mathbf{k} .

Example 5.4. Let \mathfrak{a} be an ideal. Then the sequence \mathfrak{a}_\bullet defined by $\mathfrak{a}_k = \mathfrak{a}^k$ is a graded sequence of ideals. The sequence \mathfrak{a}_\bullet is primary if and only if \mathfrak{a} is primary.

Let $\mathfrak{a}'_\bullet, \mathfrak{a}''_\bullet$ be graded sequences of ideals. Then the sequence $\mathfrak{a}_\bullet = \mathfrak{a}'_\bullet \mathfrak{a}''_\bullet$ defined by

$$\mathfrak{a}_k = \mathfrak{a}'_k \mathfrak{a}''_k,$$

is also a graded sequence of ideals which we call the *product of \mathfrak{a}'_\bullet and \mathfrak{a}''_\bullet* .

Definition 5.5. Let \mathfrak{a}_\bullet be a primary sequence of ideals. Define the function $H_{\mathfrak{a}_\bullet}$ by:

$$H_{\mathfrak{a}_\bullet}(k) = \dim_{\mathbf{k}}(R/\mathfrak{a}_k).$$

It is called the *Hilbert-Samuel function of \mathfrak{a}_\bullet* . The *Hilbert-Samuel function $H_{\mathfrak{a}}(k)$* of an ideal \mathfrak{a} is the Hilbert-Samuel function of the sequence $\mathfrak{a}_\bullet = (\mathfrak{a}, \mathfrak{a}^2, \dots)$. That is, $H_{\mathfrak{a}}(k) = \dim_{\mathbf{k}}(R/\mathfrak{a}^k)$.

Remark 5.6. It is well-known that, for sufficiently large values of k , the Hilbert-Samuel function $H_{\mathfrak{a}}$ coincides with a polynomial called the *Hilbert-Samuel polynomial of \mathfrak{a}* [Samuel-Zariski60].

Definition 5.7. Let \mathfrak{a}_\bullet be a graded sequence of ideals. The *multiplicity $e(\mathfrak{a}_\bullet)$* is defined to be:

$$e(\mathfrak{a}_\bullet) = n! \limsup_k \frac{\dim_{\mathbf{k}}(R/\mathfrak{a}_k)}{k^n}.$$

(It is not a priori clear that the limit exists.) The multiplicity $e(\mathfrak{a})$ of an ideal \mathfrak{a} is the multiplicity of its associated sequence $(\mathfrak{a}, \mathfrak{a}^2, \dots)$. That is:

$$e(\mathfrak{a}) = n! \lim_{k \rightarrow \infty} \frac{\dim_{\mathbf{k}}(R/\mathfrak{a}^k)}{k^n}.$$

The notion of multiplicity comes from the following basic example:

Example 5.8. Let \mathfrak{a} be a primary ideal in the local ring $R = \mathcal{O}_{X,p}$ of a point p in an irreducible variety X over \mathbf{k} . Let g_1, \dots, g_r be generators for the ideal \mathfrak{a} and let f_1, \dots, f_n be generic \mathbf{k} -linear combinations of the g_i . Then the multiplicity $e(\mathfrak{a})$ is equal to the intersection multiplicity at p of the hypersurfaces $H_i = \{x \mid f_i(x) = 0\}$, $i = 1, \dots, n$.

In the next section we use the material in Section 4 to give a formula for $e(\mathfrak{a}_\bullet)$ in terms of covolume of a convex region.

6. VALUATIONS AND IDEALS

Let R be an algebra over a field \mathbf{k} . Equip the group \mathbb{Z}^n with a total order respecting addition.

Definition 6.1 (Valuation). A *valuation $v : R \setminus \{0\} \rightarrow \mathbb{Z}^n$* is a function satisfying:

- (1) For all $f, g \in F$, $v(fg) = v(f) + v(g)$.
- (2) For all $f, g \in F$, $v(f + g) \geq \min(v(f), v(g))$.
- (3) For all $f \in F$ and $\lambda \neq 0 \in \mathbf{k}$, $v(\lambda f) = v(f)$.

We say that v has *one-dimensional leaves* if whenever $v(f) = v(g)$, there is $\lambda \neq 0 \in \mathbf{k}$ with $v(g - \lambda f) > v(g)$. Equivalently, v has one-dimensional leaves if the residue field of v is \mathbf{k} .

For the rest of the paper we assume R to be a local domain of dimension n , such that R is an algebra over a field \mathbf{k} isomorphic to the residue field R/\mathfrak{m} . Moreover, we assume that R has the following property:

Assumption: *There exists a valuation $v : R \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}^n$ with one-dimensional leaves, such that the image of v coincides with the whole $\mathbb{Z}_{\geq 0}^n$.*

Remark 6.2. If R is a regular local ring, in particular if R is the local ring of a smooth point on a variety, then it satisfies the above assumption.

Remark 6.3. In fact the arguments in the rest of the paper, with slight modification, work under the weaker assumption that: *the image of v consists of all the integral points in a closed strongly convex rational polyhedral cone C .* The local ring of a toroidal singularity satisfies this weaker condition. We call a singular point p toroidal if a neighborhood of p is analytically isomorphic to a neighborhood of a point in an affine toric variety.

Definition 6.4. For an ideal \mathfrak{a} in R define $\mathcal{I} = \mathcal{I}(\mathfrak{a}) \subset \mathbb{Z}_{\geq 0}^n$ by:

$$\mathcal{I} = \{v(f) \mid f \in \mathfrak{a} \setminus \{0\}\}.$$

It is easy to see that \mathcal{I} is a semigroup ideal. We call \mathcal{I} *the semigroup ideal associated to \mathfrak{a}* . Similarly, for a graded sequence of ideals \mathfrak{a}_\bullet in R , define $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet) \subset \mathbb{Z}_{\geq 0}^n$ by:

$$\mathcal{I}_k = \{v(f) \mid f \in \mathfrak{a}_k \setminus \{0\}\}.$$

\mathcal{I}_\bullet is called the *sequence of semigroup ideals associated to \mathfrak{a}_\bullet* .

Suppose $\mathbb{Z}_{\geq 0}^n$ is well-ordered with respect to $>$, e.g. a lexicographic order.

Proposition 6.5. *Let \mathcal{I} be the semigroup ideal associated to a primary ideal \mathfrak{a} in R . Then we have:*

$$\dim_{\mathbf{k}}(R/\mathfrak{a}) = \#(\mathbb{Z}_{\geq 0}^n \setminus \mathcal{I}).$$

Proof. Let $\{v_1, \dots, v_r\} \subset \mathbb{Z}_{\geq 0}^n \setminus \mathcal{I}$. Let $b_1, \dots, b_r \in R$ be such that $v(b_i) = v_i$, $i = 1, \dots, r$. We claim that no linear combination of b_1, \dots, b_r lies in \mathfrak{a} . By contradiction suppose $\sum_i c_i b_i = a \in \mathfrak{a}$. Then $v(\sum_i c_i b_i)$ is equal to $v(b_j)$ for some j . This implies that $v(b_j)$ should lie in \mathcal{I} which contradicts the choice of the v_i . Thus we conclude that $\#(\mathbb{Z}_{\geq 0}^n \setminus \mathcal{I}) \leq \dim_{\mathbf{k}}(R/\mathfrak{a})$, and hence is finite. Let B be a finite subset of R such that $v(b)$, $b \in B$ are distinct and $\{v(b) \mid b \in B\}$ coincides with $\mathbb{Z}_{\geq 0}^n \setminus \mathcal{I}$. Among the set of elements in R that are not in the span of \mathfrak{a} and B take f with minimal $v(f)$. If $v(f) = v(b)$ for some $b \in B$, then we can subtract a multiple of b from f getting an element g with $v(g) < v(f)$ which contradicts the choice of f . Similarly $v(f)$ can not lie in \mathcal{I} otherwise we can subtract an element of \mathfrak{a} from f to arrive at a similar contradiction. This shows that the set of images of elements of B in R/\mathfrak{a} is a \mathbf{k} -vector space basis for R/\mathfrak{a} which proves the proposition. \square

Corollary 6.6. *If \mathfrak{a}_\bullet is a primary sequence of ideals in R then $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet)$ is a primary sequence of semigroup ideal, and moreover:*

$$e(\mathfrak{a}_\bullet) = e(\mathcal{I}_\bullet).$$

Definition 6.7. To the sequence of ideals \mathfrak{a}_\bullet we associate a $\mathbb{R}_{\geq 0}^n$ -convex region $\Gamma(\mathfrak{a}_\bullet)$, which is the convex region $\Gamma(\mathcal{I}_\bullet)$ associated to the sequence of semigroup ideals $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet)$. The convex region $\Gamma(\mathfrak{a}_\bullet)$ depends on the choice of the valuation

v. By definition the convex region $\Gamma(\mathfrak{a})$ associated to an ideal \mathfrak{a} is the convex region associated to the sequence of ideals $(\mathfrak{a}, \mathfrak{a}^2, \mathfrak{a}^3, \dots)$.

Theorem 6.8. *Let \mathfrak{a}_\bullet be a primary sequence of ideals in R . Then:*

$$e(\mathfrak{a}_\bullet) = n! \operatorname{covol}(\Gamma(\mathfrak{a}_\bullet)).$$

In particular, if \mathfrak{a} is a primary ideal we have $e(\mathfrak{a}) = n! \operatorname{covol}(\Gamma(\mathfrak{a}))$.

The following superadditivity follows from Proposition 4.9.

Proposition 6.9. *Let $\mathfrak{a}'_\bullet, \mathfrak{a}''_\bullet$ be two sequences of ideals in R . We have:*

$$\Gamma(\mathfrak{a}'_\bullet) + \Gamma(\mathfrak{a}''_\bullet) \subset \Gamma(\mathfrak{a}'_\bullet \mathfrak{a}''_\bullet).$$

From Proposition 6.9 and Theorem 2.4 we readily obtain:

Corollary 6.10. *(Brunn-Minkowski for multiplicities) Let $\mathfrak{a}'_\bullet, \mathfrak{a}''_\bullet$ be two primary sequences of ideals in R . Then:*

$$e(\mathfrak{a}'_\bullet)^{1/n} + e(\mathfrak{a}''_\bullet)^{1/n} \geq e(\mathfrak{a}'_\bullet \mathfrak{a}''_\bullet)^{1/n}.$$

7. CASE OF MONOMIAL IDEALS AND NEWTON POLYHEDRA

In this section we discuss the case of monomial ideals. It is related to the classical notion of Newton polyhedron of a power series in n variables. Applied to a monomial ideal, our Theorem 6.8 recovers the local version of celebrated theorem of Bernstein-Kushnirenko [Arnold-Varchenko-Guseinzade85, Section 12.7].

Let R be the local ring of an affine toric variety at its torus fixed point. The algebra R can be realized as follows. Let $C \subset \mathbb{R}^n$ be a strongly convex rational polyhedral cone with apex at the origin, that is, C is a convex cone generated by a finite number of rational vectors and it does not contain any line through the origin. Consider the semigroup algebra over \mathbf{k} of the semigroup of integral points $C \cap \mathbb{Z}^n$. In other words, consider the algebra of Laurent polynomials consisting of all the f of the form $f = \sum_{\alpha \in C \cap \mathbb{Z}^n} c_\alpha x^\alpha$, where we have used the shorthand notation $x = (x_1, \dots, x_n)$, $\alpha = (a_1, \dots, a_n)$ and $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$. Let R be the localization of this Laurent polynomial algebra at the maximal ideal \mathfrak{m} generated by non-constant monomials. (Similarly instead of R we can take its completion at the maximal ideal \mathfrak{m} which is an algebra of power series.)

Definition 7.1. Let \mathfrak{a} be a monomial ideal in R , that is, an ideal generated by monomials. To \mathfrak{a} we can associate a semigroup ideal $\mathcal{I}(\mathfrak{a})$ by

$$\mathcal{I}(\mathfrak{a}) = \{\alpha \mid x^\alpha \in \mathfrak{a}\}.$$

One verifies that since \mathfrak{a} is an ideal generated by monomials, $\mathcal{I}(\mathfrak{a})$ is a semigroup ideal in $C \cap \mathbb{Z}^n$ generated by the exponents of monomial generators of \mathfrak{a} . The convex hull of the semigroup ideal $\mathcal{I}(\mathfrak{a})$ is usually called the *Newton polyhedron* of the monomial ideal \mathfrak{a} . It is a convex unbounded polyhedron in C , moreover it is a C -convex region. The *Newton diagram* of \mathfrak{a} is the union of bounded faces of the Newton polyhedron.

Consider a total order on \mathbb{Z}^n which respects addition and such that the semigroup $C \cap \mathbb{Z}^n$ is well-ordered. One defines a valuation v on the algebra R with values in $C \cap \mathbb{Z}^n$ as follows. For $f = \sum_{\alpha \in C \cap \mathbb{Z}^n} c_\alpha x^\alpha$ let

$$v(f) = \min\{\alpha \mid c_\alpha \neq 0\}.$$

One verifies that this is a valuation, moreover it extends to the field of rational functions (respectively Laurent series) by defining $v(f/g) = v(f) - v(g)$ for Laurent polynomials (respectively power series) f and g . The following is easy to prove (using similar arguments as in [Khovanskii92]).

Proposition 7.2. *The convex region $\Gamma(\mathfrak{a})$ associated to the monomial ideal \mathfrak{a} and the valuation v (Definition 6.7) coincides with the Newton polyhedron of \mathfrak{a} defined above.*

We also make the following important observation that the Newton polyhedron is additive with respect to the product of ideals.

Proposition 7.3. *Let $\mathfrak{a}_1, \mathfrak{a}_2$ be monomial ideals in R . Then $\mathcal{I}(\mathfrak{a}_1\mathfrak{a}_2) = \mathcal{I}(\mathfrak{a}_1) + \mathcal{I}(\mathfrak{a}_2)$. It follows that*

$$\Gamma(\mathfrak{a}_1\mathfrak{a}_2) = \Gamma(\mathfrak{a}_1) + \Gamma(\mathfrak{a}_2).$$

From Proposition 7.2, Proposition 7.3 and Theorem 6.8 we readily obtain the following.

Theorem 7.4. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be primary monomial ideals in R . Then the mixed multiplicity $e(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ is given by:*

$$e(\mathfrak{a}_1, \dots, \mathfrak{a}_n) = n! \operatorname{covol}(\Gamma(\mathfrak{a}_1), \dots, \Gamma(\mathfrak{a}_n)).$$

One knows that the mixed multiplicity of the primary ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ gives the intersection multiplicity, at the origin, of hypersurfaces $H_i = \{x \mid f_i(x) = 0\}$, $i = 1, \dots, n$, where each f_i is a generic element from the ideal \mathfrak{a}_i . Theorem 7.4 then gives the following corollary.

Theorem 7.5. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be primary ideals in R . Consider a system of equations $f_1(x) = \dots = f_n(x) = 0$ where each f_i is a generic element from the ideal \mathfrak{a}_i . Then the intersection multiplicity at the origin of this system is equal to $n! \operatorname{covol}(\Gamma(\mathfrak{a}_1), \dots, \Gamma(\mathfrak{a}_n))$.*

Remark 7.6. (i) When R is the algebra of polynomials $\mathbf{k}[x_1, \dots, x_n]_{(0)}$ localized at the origin (or the algebra of power series localized at the origin), i.e. the case corresponding to the local ring of a smooth affine toric variety, Theorem 7.5 is the local version of the classical Bernstein-Kushnirenko theorem ([Arnold-Varchenko-Guseinzade85, Section 12.7]).

(ii) As opposed to the proof above, the original proof of the Kushnirenko theorem is quite involved.

(iii) Theorem 7.5 (in the general case of local ring of an affine toric variety) has been known to the second author since the early 90's (cf. [Khovanskii92]), although as far as the authors know it has not been published.

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